Objective

Students will be able to find linear factors (rational zeros) of polynomial expressions.

Quiz (5.6, 5.7, 5.9) on Friday!

The Rational Zero Theorem

The polynomial function $f(x) = 64x^3 + 152x^2 - 62x - 105$ has $-\frac{5}{2}, -\frac{3}{4}, and\, \frac{7}{8}$ as its zeros.

Notice that the numerators of these zeros (-5, -3, and 7) are factors of the constant term, -105. Also notice that the denominators (2, 4, and 8) are factors of the leading coefficient, 64.

The Rational Zero Theorem

If $f(x) = a_n x^n + \cdots + a_1 x + a_0$ has *integer* coefficients, then every rational zero of f has the following form:

$$\frac{p}{q} = \frac{\text{factor of constant term } a_0}{\text{factor of leading coefficient } a_n}$$

List the possible rational zeros of f using the rational zero theorem.

$$f(x) = 4x^4 - x^3 - 3x^2 + 9x - 10$$

Factors of the constant term: $\pm 1, \pm 2, \pm 5, \pm 10$

Factors of the leading coefficient: $\pm 1, \pm 2, \pm 4$

Possible rational zeros:

$$\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{5}{1}, \pm\frac{10}{1}, \pm\frac{1}{2}, \pm\frac{2}{2}, \pm\frac{5}{2}, \pm\frac{10}{2}, \pm\frac{1}{4}, \pm\frac{2}{4}, \pm\frac{5}{4}, \pm\frac{10}{4}$$

Simplified list of possible rational zeros:

$$\pm 1, \pm 2 \pm 5, \pm 10, \pm\frac{1}{2}, \pm\frac{5}{2}, \pm\frac{1}{4}, \pm\frac{5}{4}$$

List the possible rational zeros of *f* using the rational zero theorem.

1) $f(x) = x^3 + 2x^2 - 11x + 12$

$$\pm 1, \pm 2 \pm 3, \pm 4, \pm 6, \pm 12$$

2) $f(x) = 2x^3 + 3x^2 + 4x - 6$

$$\pm 1, \pm 2 \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$$

1) Find all real zeros of $f(x) = x^3 - 8x^2 + 11x + 20$

**Leading coefficient is one

Step 1: List all possible rational zeros $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$

Step 2: Test these zeros using synthetic division

Test x = 1:

$$\begin{array}{r|rrrr} 1 & 1 & -8 & 11 & 20 \\ & & 1 & -7 & 4 \\ \hline & 1 & -7 & 4 & 24 \end{array}$$

1 is not a zero

Test x = -1:

$$\begin{array}{r|rrrr} -1 & 1 & -8 & 11 & 20 \\ & & -1 & 9 & -20 \\ \hline & 1 & -9 & 20 & 0 \end{array}$$

-1 is a zero

Because -1 is a zero of *f*, we can write $f(x) = (x+1)(x^2 - 9x + 20)$

Step 3: Factor the trinomial in *f(x)* and use the factor theorem to solve

$$f(x) = (x+1)(x^2 - 9x + 20) \;\; = (x+1)(x-4)(x-5)$$

The zeros of *f* are -1, 4, and 5.

**Degree of polynomial is 3, so there should be at most 3 real zeros

2) Find all real zeros of $f(x) = x^3 - 4x^2 - 15x + 18$

**Leading coefficient is one

Step 1: List all possible rational zeros $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$

Step 2: Test these zeros using synthetic division

Test x = 1:

$$\begin{array}{r|rrrr} 1 & 1 & -4 & -15 & 18 \\ & & 1 & -3 & -18 \\ \hline & 1 & -3 & -18 & 0 \end{array}$$

1 is a zero

Because 1 is a zero of *f*, we can write $f(x) = (x-1)(x^2 - 3x - 18)$

Step 3: Factor the trinomial in *f(x)* and use the factor theorem to solve

$$f(x) = (x-1)(x^2 - 3x - 18) \quad = (x-1)(x+3)(x-6)$$

The zeros of *f* are 1, -3, and 6.

Homework

p. 374: 4, 5, 8, 17, 18

LISTING RATIONAL ZEROS **List the possible rational zeros of the function using the rational zero theorem.**

4. $g(x) = x^3 - 4x^2 + x - 10$

5. $f(x) = 2x^4 + 6x^3 - 7x + 9$

8. $f(x) = 3x^4 + 5x^3 - 3x + 42$

FINDING REAL ZEROS **Find all real zeros of the function.**

17. $f(x) = x^4 + 2x^3 - 9x^2 - 2x + 8$

18. $g(x) = x^4 - 16x^2 - 40x - 25$

***only need to find the *real* zeros

Objective

Students will be able to find linear factors (rational zeros) of polynomial expressions.

Quiz (5.6, 5.7, 5.9) on Friday!

Find all real zeros of polynomial functions.

If you are given a polynomial function that is greater than a cubic function, you need to keep using synthetic division until you get down to a function that is factorable (usually a quadratic function).

Find all real zeros of $f(x) = 10x^4 - 11x^3 - 42x^2 + 7x + 12$

Step 1: List all possible rational zeros

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{4}{5}, \pm\frac{6}{5}, \pm\frac{12}{5}, \pm\frac{1}{10}, \pm\frac{3}{10}$$

Step 2: Graph the function; Choose reasonable values from the list above to check using the graph of the function.

For f, the values

$$x = -\frac{3}{2}, x = -\frac{1}{2}, x = \frac{3}{5}, x = \frac{12}{5}$$

are reasonable based on the graph shown at the right.

Step 3: Check the values using synthetic division until a zero is found.

Test x = -3/2:

-3/2	10	-11	-42	7	12
		-15	39	9/2	-69/4
	10	-26	-3	23/2	-21/4

-3/2 is not a zero

Test x = -1/2:

-1/2	10	-11	-42	7	12
		-5	8	17	-12
	10	-16	-34	24	0

-1/2 is a zero

Step 4: Factor out a binomial using the result of the synthetic division.

$$f(x) = (x + \frac{1}{2})(10x^3 - 16x^2 - 34x + 24)$$

Write as a product of factors

$$f(x) = (x + \frac{1}{2})(2)(5x^3 - 8x^2 - 17x + 12)$$

Factor 2 out of the second factor

$$f(x) = (2x + 1)(5x^3 - 8x^2 - 17x + 12)$$

Multiply the first factor by 2

Step 5: Repeat the steps about for $g(x) = 5x^3 - 8x^2 - 17x + 12$. Any zero of g will also be a zero of f.

$$x = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{4}{5}, \pm\frac{6}{5}, \pm\frac{12}{5}$$

The graph of g shows that $\frac{3}{5}$ may be a zero. Synthetic division shows that $\frac{3}{5}$ *is* a zero and $g(x) = \left(x - \frac{3}{5}\right)(5x^2 - 5x - 20) = (5x - 3)(x^2 - x - 4)$.

It follows that:

$$f(x) = (2x + 1) \cdot g(x) = (2x + 1)(5x - 3)(x^2 - x - 4)$$

Step 6: Find the remaining zeros of *f* by solving $x^2 - x - 4 = 0$.

Unfactorable; have to use quadratic formula

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-4)}}{2(1)} \qquad = \frac{1 \pm \sqrt{17}}{2}$$

The real zeros of *f* are

$$-\frac{1}{2}, \frac{3}{5}, \frac{1+\sqrt{17}}{2}, \frac{1-\sqrt{17}}{2}$$

**Degree of polynomial is 4, so there should be at most 4 real zeros

Homework p. 374: 11, 19, 21, 35

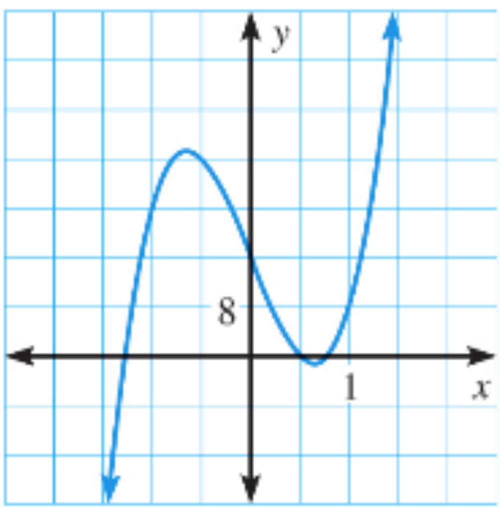
FINDING REAL ZEROS **Find all real zeros of the function.**

11. $f(x) = x^3 - 12x^2 + 35x - 24$

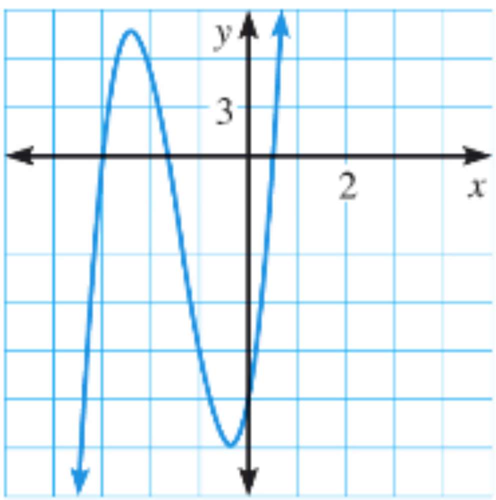
35. $h(x) = 2x^5 + 5x^4 - 3x^3 - 2x^2 - 5x + 3$

ELIMINATING POSSIBLE ZEROS **Use the graph to shorten the list of possible rational zeros of the function. Then find all real zeros of the function.**

19. $f(x) = 4x^3 - 20x + 16$



21. $f(x) = 6x^3 + 25x^2 + 16x - 15$



Objective

Students will be able to find the number of zeros of polynomial functions, including repeating zeros and imaginary solutions.

Students will also be able to write polynomial functions using the least degree that has rational coefficients, a leading coefficient of 1, and given zeros.

Quiz (5.6, 5.7, 5.9) on **Tuesday**!

The Fundamental Theorem of Algebra

The equation $x^3 - 5x^2 - 8x + 48 = 0$, which becomes $(x + 3)(x - 4)^2 = 0$ when factored, has only two distinct solutions: -3 and 4. Because the factor $x - 4$ appears twice, however, you can count the solution 4 twice. So, with 4 counted as a repeated solution, this *third*-degree equation has *three* solutions: -3, 4, and 4.

The Fundamental Theorem of Algebra

Theorem: If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has at least one solution in the set of complex numbers.

Corollary: If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has exactly n solutions provided each solution repeated twice is counted as 2 solutions, each solution repeated three times is counted as 3 solutions, and so on.

Find <u>all</u> zeros of

$f(x) = x^5 - 4x^4 + 4x^3 + 10x^2 - 13x - 14$

Find all rational zeros of *f* and solve using synthetic division.

Because *f* is a polynomial function of degree 5, it has 5 zeros.

The possible zeros are ± 1, ± 2, ± 7, ± 14.

Test x = -1:

$$\begin{array}{r|rrrrrr} -1 & 1 & -4 & 4 & 10 & -13 & -14 \\ & & -1 & 5 & -9 & -1 & 14 \\ \hline & 1 & -5 & 9 & 1 & -14 & 0 \end{array}$$

-1 is a zero

$f(x) = (x + 1)(x^4 - 5x^3 + 9x^2 + x - 14)$

Test x = -1:

$$\begin{array}{r|rrrrr} -1 & 1 & -5 & 9 & 1 & -14 \\ & & -1 & 6 & -15 & 14 \\ \hline & 1 & -6 & 15 & -14 & 0 \end{array}$$

-1 is a zero

$f(x) = (x + 1)^2(x^3 - 6x^2 + 15x - 14)$

Test x = 2:

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 15 & -14 \\ & & 2 & -8 & 14 \\ \hline & 1 & -4 & 7 & 0 \end{array}$$

2 is a zero

$f(x) = (x + 1)^2(x - 2)(x^2 - 4x + 7)$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(7)}}{2(1)} = 2 \pm i\sqrt{3}$$

The zeros are

$$-1, -1, 2, 2 + i\sqrt{3}, 2 - i\sqrt{3}$$

Complex Conjugates

Notice in the warm up, how the zeros -2 + 2*i*√2 and -2 – 2*i*√2 are complex conjugates.

Complex Conjugates Theorem

If f is a polynomial function with real coefficients, and $a + bi$ is an imaginary zero of f, then $a - bi$ is also a zero of f.

Irrational Conjugates Theorem

Suppose f is a polynomial function with rational coefficients, and a and b are rational numbers such that $\sqrt{b}$ is irrational. If $a + \sqrt{b}$ is a zero of f, then $a - \sqrt{b}$ is also a zero of f.

Write the polynomial function *f* of least degree that has rational coefficients, a leading coefficient of 1, and also has 3 and 2 + √5 as zeros.

Because the coefficients are rational and 2 + √5 is a zero, 2 – √5 must be a zero by the irrational conjugates theorem. Use the three zeros and the factor theorem to write *f(x)* as a product of three factors.

Write f(x) in factored form $f(x)=(x-3)[x-(2+\sqrt{5})][x-(2-\sqrt{5})]$

Regroup terms $=(x-3)[(x-2)-\sqrt{5}][(x-2)+\sqrt{5}]$

Multiply (FOIL) $=(x-3)[(x-2)^2+\sqrt{5}(x-2)-\sqrt{5}(x-2)-\sqrt{5}(\sqrt{5})]$

Expand/Distribute $=(x-3)[x^2-4x+4+\cancel{x\sqrt{5}}-\cancel{2\sqrt{5}}-\cancel{x\sqrt{5}}+\cancel{2\sqrt{5}}-5]$

Simplify $=(x-3)(x^2-4x-1)$

Multiply $=x^3-4x^2-x-3x^2+12x+3$

Combine like terms $=x^3-7x^2+11x+3$

$$\boxed{f(x)=x^3-7x^2+11x+3}$$

Homework

p. 383: 7-11 (odds), 15, 24, 29

NUMBER OF SOLUTIONS OR ZEROS **Identify the number of solutions or zeros.**

7. $g(s) = 12s^7 - 9s^6 + 4s^5 - s^3 - 20s + 50$

9. ★ **MULTIPLE CHOICE** How many zeros does the function $f(x) = 16x - 22x^3 + 6x^6 + 19x^5 - 3$ have?

(A) 1 (B) 3 (C) 5 (D) 6

FINDING ZEROS **Find all zeros of the polynomial function.**

11. $f(x) = x^4 + 5x^3 - 7x^2 - 29x + 30$

15. $f(x) = x^4 + x^3 + 2x^2 + 4x - 8$

WRITING POLYNOMIAL FUNCTIONS **Write a polynomial function *f* of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros.**

24. $2, -i, i$

29. $-4, 1, 2 - \sqrt{6}$